Introduction

- Frequency response techniques provide another perspective for design of feedback control systems via gain adjustment and compensation.
- Has advantages in following situations:
  1. Modelling Transfer functions from physical data.
  2. Designing lead compensators to meet steady-state error and transient response requirement.
  3. Finding stability of nonlinear systems.
  4. Settling ambiguities when sketching root locus.

Modelling Example

- The HP 35670A Dynamic Signal Analyzer can be used to obtain frequency response data from a physical system.
- This information can be used to analyze, design, or create a mathematical model for the system.

Figure 10.1.

Concept of Frequency Response

- In steady state, sinusouidal inputs to a system generate a sinsusoidal response at the same frequency.

- The response differs in amplitude and phase angle from input.
- Differences are a function of frequency (ω).
- Consider:

$$\mathcal{L}\{K_1 e^{-at} cos\omega t + K_2 e^{-at} sin\omega t\} = \frac{K_1(s+a) + K_2\omega}{(s+a)^2 + \omega^2}$$

- Can show that

$$e^{-at}[2\alpha \, \cos\omega t + 2\beta \, \sin\omega t]$$
$$= K_3 e^{-at} \cos(\omega t + \phi)$$

where $\phi = -\tan^{-1}(\frac{\beta}{\alpha})$ and $K_3 = \sqrt{(2\alpha)^2 + (2\beta)^2}$

Phasors

- Phasors are a convenient way to represent sinusoids.
- Phasors are complex numbers whose magnitude is the amplitude of the sinusoid, and angle is the phase angle of the sinusoid.
- For example, sinusoid $M_1 \cos(\omega t + \phi_1)$ would be the phasor $M_1 \angle \phi_1$, where frequency, ω, is implicit.

Concept of Frequency Response: e.g.

- As a system causes both the magnitude and phase angle of a sinusoidal input to change, it can be thought of as a phasor.

- Magnitude and phase angle of system chosen so that product of input phasor and system produce output phasor.

- Part (a) is the physical system, (b) the transfer function, and (c) the input and output waveforms.

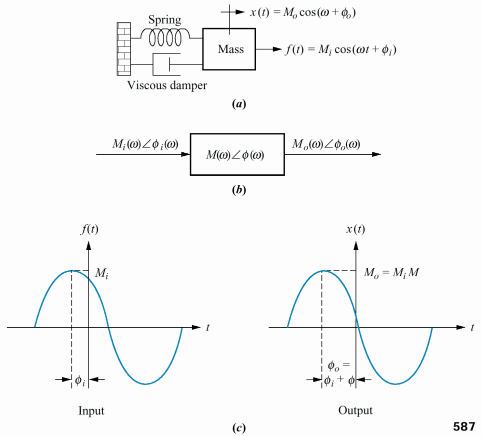


Figure: 8.4

Concept of Frequency Response: e.g. II

- We represent the input as phasor $M_i(\omega)\angle\phi_i(\omega)$, and output as phasor $M_o(\omega)\angle\phi_o(\omega)$.

- We assume that the system is represented by phasor $M(\omega)\angle\phi(\omega)$.

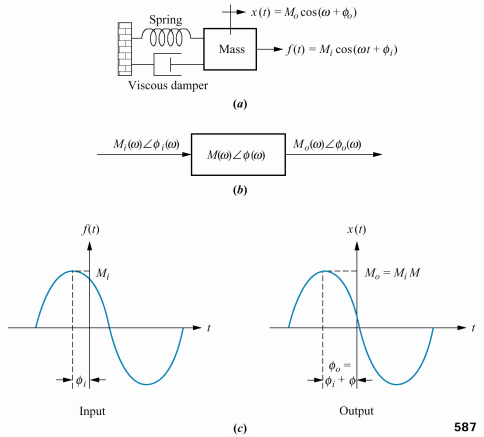


Figure: 8.4

Concept of Frequency Response: e.g. III

- The steady-state output phasor is thus:

$$M_o(\omega)\angle\phi_o(\omega) = M_i(\omega)M(\omega)\angle[\phi_i(\omega) + \phi(\omega)] \tag{1}$$

- We can thus represent system function as:

$$M(\omega) = \frac{M_o(\omega)}{M_i(\omega)} \tag{2}$$

$$\phi(\omega) = \phi_o(\omega) - \phi_i(\omega) \tag{3}$$

- We call $M(\omega)$ the magnitude frequency response and $\phi(\omega)$ the phase frequency response.

- We call the combination of the two the frequency response.
- The frequency response of a system whose transfer function is $G(s)$ is $G(jw) = G(s)|_{s\rightarrow jw}$

Plotting Frequency Response

- We plot $G(jw) = M(\omega)\angle\phi(\omega)$ as a function of frequency, with a separate plot for magnitude and phase.
- Magnitude is plotted in decibels (dB) versus $\log\omega$, where db $= 20\log M$.
- Phase curve is plotted as phase angle versus $\log\omega$.

What the hell is a decibel?

In systems we are often interested in how a system affects a signal from its input to its output. One important measure is the power gain from the input to the output $\frac{P_{out}}{P_{in}}$. A *bel* is the $\log_{10}$ of this ratio. A *decibel* (dB) is one tenth of a bel. Thus the power gain in decibels is

$$10 \log_{10} \frac{P_{out}}{P_{in}} = 10 \log_{10} \frac{P_Y}{P_U}$$

for our system $G(s) = \frac{Y(s)}{U(s)}$.

For a voltage or a current, power varies as the square of the amplitude of the signal. E.g.,

$$\frac{P_{out}}{P_{in}} = \frac{v_{out}^2/R}{v_{in}^2/R} = \left(\frac{v_{out}}{v_{in}}\right)^2 \text{ or } \frac{P_{out}}{P_{in}} = \frac{i_{out}^2 R}{i_{in}^2 R} = \left(\frac{i_{out}}{i_{in}}\right)^2$$

Thus the power gain $\frac{P_Y}{P_U}$ in decibels of $G(s)$ for sinusoidal input at frequency ω_1 is:

$$10 \log_{10} \left(\frac{Y(j\omega_1)}{U(j\omega_1)}\right)^2 = 20 \log_{10} |G(j\omega_1)| \text{ dB}$$

System Bandwidth ω_{BW}

The *bandwidth* of the system (ω_{BW}) is defined to be the maximum frequency at which the system will satisfactorily track a sinusoidal input.

By "satisfactory" tracking we roughly mean that the power from the input to the output is reduced by no more than $\frac{1}{2}$.

Since power varies as the square of the amplitude of the signal we have at the bandwidth frequency ω_{BW}:

$$\frac{1}{2} = \frac{P_Y}{P_U} = \frac{|Y(jw_{BW})|^2}{|U(jw_{BW})|^2} = |G(jw_{BW})|^2$$

So we must have $|G(jw_{BW}| = \frac{1}{\sqrt{2}} \cong 0.707$. Measuring the power gain in decibels (dB) we have

$$\begin{aligned} 20\log_{10}|G(jw_{BW})| &= 20\log_{10}(\frac{1}{\sqrt{2}}) \\ &= -3 \text{ dB} \end{aligned}$$

Plotting Frequency Response: e.g.

- Find analytical expression for magnitude and phase angle frequency response for system $G(s) = \frac{1}{(s+2)}$ and plot both diagrams.

- We start by setting $s = j\omega$ and simplifying.

$$G(j\omega) = \frac{1}{(j\omega + 2)} \times \frac{(2 - j\omega)}{(2 - j\omega)}$$

$$= \frac{(2 - j\omega)}{(2j\omega - (j\omega)^2 + 4 - 2j\omega)} = \frac{(2 - j\omega)}{(\omega^2 + 4)}$$

- We thus have:

$$M(\omega) = \sqrt{\frac{4}{(\omega^2 + 4)^2} + \frac{\omega^2}{(\omega^2 + 4)^2}} = \frac{1}{\sqrt{(\omega^2 + 4)}}$$

$$\phi(\omega) = \tan^{-1}\left(\frac{-\omega}{2}\right)$$

Plotting Frequency Response: e.g. III

- Plot of magnitude and phase angle frequency response for system $G(s) = \frac{1}{(s+2)}$.

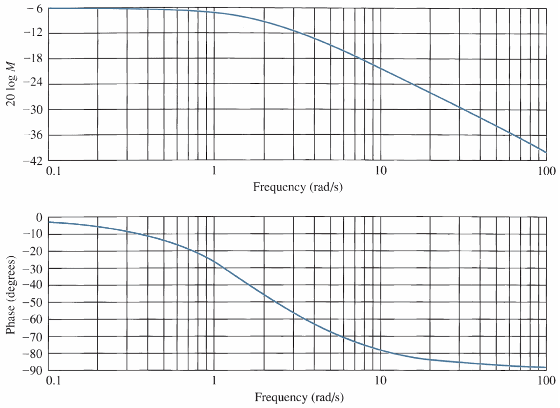


Figure 10.4.

Asymptotic Approximations: Bode Plots

- Log-magnitude and phase frequency response curves plotted as function of $\log \omega$ are called Bode plots or Bode diagrams.
- Can be sketched easily by approximating as a sequence of straight lines.
- Consider transfer function:

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- Magnitude frequency response is thus:

$$|G(j\omega)| = \left. \frac{K|(s+z_1)|\,|(s+z_2)|\cdots|(s+z_k)|}{|s^m|\,|(s+p_1)|\,|(s+p_2)|\cdots|(s+p_n)|} \right|_{s\rightarrow j\omega}$$

Asymptotic Approximations: Bode Plots II

- As $\log(ab) = \log(a) + \log(b)$ and $\log(a/b) = \log(a) - \log(b)$, converting to dB gives

$$\begin{aligned} 20\log|G(j\omega)| &= 20\log K + 20\log|(s+z_1)| + 20\log|(s+z_2)| \\ &\quad + \cdots + 20\log|(s+z_k)| - 20\log|s^m| \\ &\quad - 20\log|(s+p_1)| - \cdots - 20\log|(s+p_n)|_{s\to j\omega} \end{aligned}$$

- The phase frequency response would be the summation of the phase frequency response curves of the zero terms minus the summation of the phase frequency response curves of the pole terms.

Bode Plots for $G(s) = (s + a)$

- Letting $s = j\omega$ gives

$$G(j\omega) = (j\omega + a)$$

- At low frequencies when ω approaches zero, we get $G(j\omega) \approx a$ with magnitude response $M = |G(j\omega)|$ is:

$$20 \log M = 20 \log a$$

- At high frequencies with $\omega \gg a$, we get $G(j\omega) \approx j\omega = \omega\angle 90^o$ where

$$20 \log M = 20 \log \omega$$

- If we plot $20 \log M = 20 \log \omega$ against $\log \omega$, we get a straight line $y = 20x$ where $y = 20 \log M$ and $x = \log \omega$.

- i.e. the line increases at rate of 20dB/decade.

Bode Plots for $G(s) = (s + a)$: II

- We know what happens when $\omega >> a$ or $a >> \omega$.
- What happens when $\omega = a$?

we get

$$
\begin{aligned}
20 \log M &= 20 \log(|a + ja|) \\
&= 20 \log \sqrt{a^2 + a^2} \\
&= 20 log(a\sqrt{2}) \\
&= 20 \log(a) + 20 \log(\sqrt{2}) \\
&= 20 \log(a) + 3.01
\end{aligned}
$$

Bode Plots for $G(s) = (s + a)$: II

- To draw magnitude approximation, start with a horizontal line at $20 \log a$, and at frequency a, switch to a line with slope 20db/decade.
- We call the straight line approximations asymptotes.
- We call a the break frequency because it is where we switch from the low-frequency asymptote to the high-frequency asymptote.

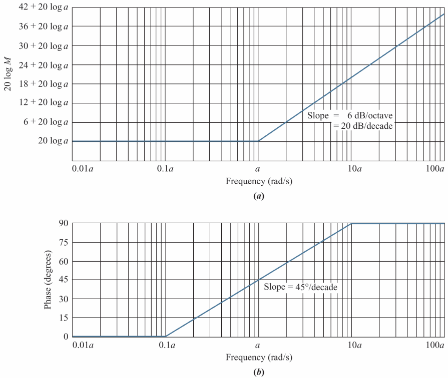


Figure 10.6

Bode Plots for $G(s) = (s + a)$: Phase

- At beakpoint $G(j\omega) = (j\omega + a) = a + ja$ which gives us a phase of 45^o.
- At low frequencies, we have $G(j\omega) \approx a$, thus a phase of 0^o.
- At high frequencies we have $G(j\omega) \approx jw$, thus a phase of 90^o.
- To draw phase curve:
  1. Start graph at $0.01a$ at phase of zero.
  2. At $0.1a$, switch to line with slope of $+45^o$ per decade.
  3. At $10a$, switch to horizontal line at 90^o.

Bode Plots: Normalizing

- To make it easier to compare plots with different break points, it's common to normalize the magnitude and scale the frequency.

- This will give all $G(s) = (s + a)$ magnitude plots a low frequency value of 0db at a unity break frequency.

- To normalize, we take $G(s) = a(\frac{s}{a} + 1)$, take our new frequency variable to be $s_1 = \frac{s}{a}$, and divide the magnitude by a.

- This gives a normalized and scaled function of $G(s_1) = (s_1 + 1)$.

Bode Plots: Normalizing - Magnitude

- Actual curve is never more than 3.01 dB from asymptotes.

- This occurs at break frequency.

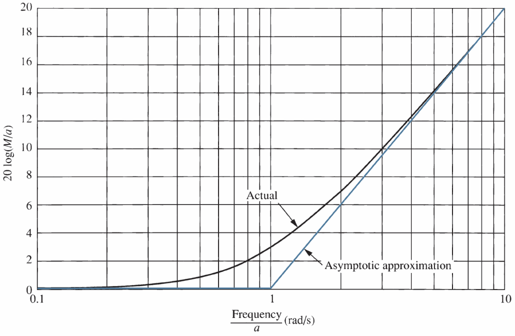


Figure 10.7.

Bode Plots: Normalizing - Phase

- Phase curve is never more than 5.71^o different from the asymptotes.

- This occurs at the decades above and below the break frequency.

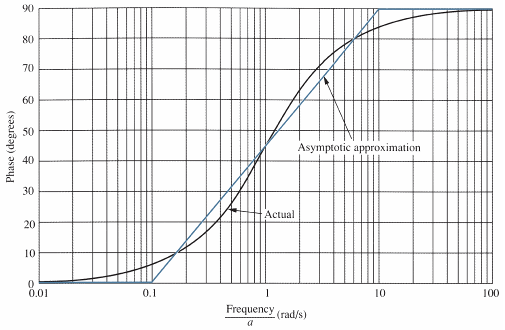


Figure 10.8.

Bode Plots for $G(s) = 1/(s+a)$

- At low frequencies when ω approaches zero, we get $G(j\omega) \approx \frac{1}{a}$ with magnitude response

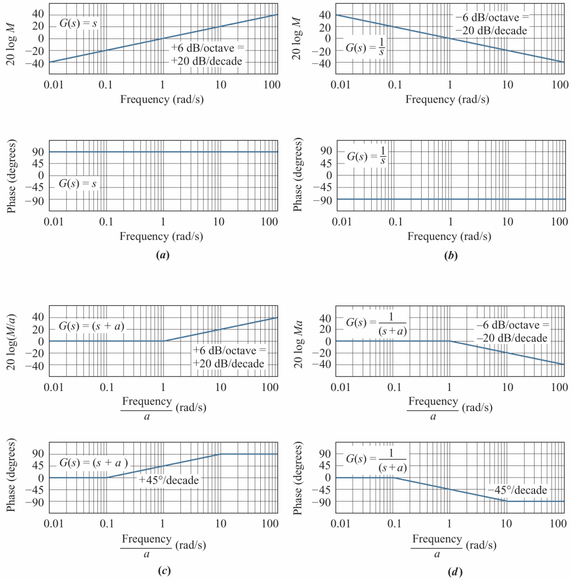
$$20\log M = 20\log\frac{1}{a} = 20\log a^{-1} = -20\log a$$

- Plot is constant until break frequency a rad/s is reached, then we use high-frequency asymptote.
- Let ω approach ∞, gives $G(j\omega) \approx \frac{1}{j\omega} = \frac{1}{\omega}\angle - 90^o$ where

$$20\log M = 20\log\frac{1}{\omega} = 20\log 1 - 20\log\omega = -20\log\omega$$

- After break frequency, plot decreases at rate of 20dB/decade.
- Phase is negative of previous function. Starts at 0^o line, then switch to -45^o/decade slope at $0.1a$. At $10a$, switch to -90^o line.

Bode Plots for $G(s)$



(a) $G(s) = s$

(b) $G(s) = \frac{1}{s}$

(c) $G(s) = s + a$

(d) $G(s) = \frac{1}{s+a}$

Figure 10.9.

Bode Plots for $G(s) = s$ and $G(s) = 1/s$

- For $G(s) = s$, we only have high-frequency asymptote with magnitude $20 \log \omega$.
- This is a straightline with $+20$db/dec slope, and equal to 0db at $\omega = 1$.
- Phase is constant at $+90^o$.
- For $G(s) = 1/s$, we get a straight line with slope -20db/dec slope, and equal to 0db at $\omega = 1$.
- Phase is constant at -90^o.

Drawing Bode Plots: Magnitude

1. Determine initial slope:
   - If contains pole or zero at origin, then determine net slope due to items at origin.
   - Otherwise, we start with a horizontal line.
2. Determine leftmost starting value.
   - If contains pole or zero at origin, then determine magnitude at $0.1a$, where a is smallest break frequency.
   - Otherwise, determine magnitude at s $= 0$.
3. At each break frequency, increase slope by $+20$dB/dec for a zero and -20dB/dec if break frequency corresponds to a pole.
4. Effect of gain K is to move the magnitude curve up $(K > 1)$ or down $(K < 1)$ by amount $20 \log K$.
5. To draw graph, take $K = 1$ unless gain is specified.

Drawing Bode Plots: Phase

1. Determine leftmost starting value.
   - If contains poles or zeros at origin, then determine net phase by adding that of the poles or zeros at origin.
   - Otherwise, phase starts at 0 degrees.
2. Start graph at $0.1a$, where a is the smallest break frequency.
3. For each zero at break frequency a:
   - At $0.1a$, increase slope by 45^o/decade.
   - At $10a$, decrease slope by 45^o/decade.
4. For each pole at break frequency a:
   - At $0.1a$, decrease slope by 45^o/decade.
   - At $10a$, increase slope by 45^o/decade.
5. Gain K has no effect on phase curve.

Bode Plots for Ratio of First-Order Factors

- Draw bode plots for system where

$$G(s) = \frac{K(s+3)}{s(s+1)(s+2)}$$

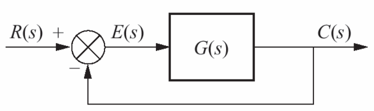
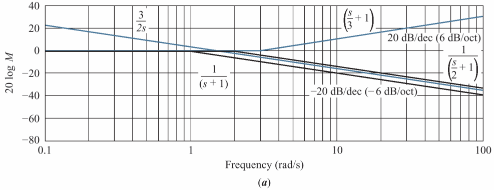


Figure 10.10.

Bode Plots for Ratio of First-Order Factors - Magnitude

- See notes on board in class & Matlab file for details. Bode plot is of *open loop* transfer function!



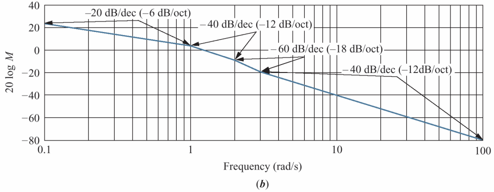


Figure 10.11.

Bode Plots for Ratio of First-Order Factors - Phase

- See notes on board in class for details

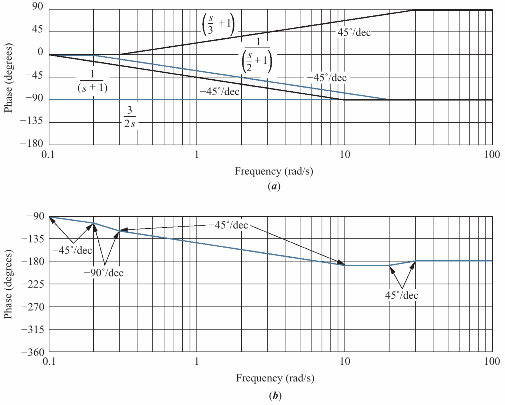


Figure 10.12.

Bode Plots for $G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$

- When ω approaches zero, we get $G(j\omega) \approx \omega_n^2 = \omega_n^2 \angle 0^o$ with magnitude response

$$20 \log M = 20 \log \omega_n^2$$

- At high frequencies, we get $G(j\omega) \approx (j\omega)^2 = -\omega^2$ $= \omega^2 \angle 180^o$ where

$$20 \log M = 20 \log \omega^2 = 40 \log \omega$$

- This is a straight line with a +40dB/dec slope.

- We note that when $\omega = \omega_n$ the high and low frequency approximations are equal, thus ω_n is the break frequency.

Bode Plots for $G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$: II

- To draw magnitude approximation, start with a horizontal line at $20 \log \omega_n^2$, and at frequency ω_n, switch to a line with slope $+40$db/decade.

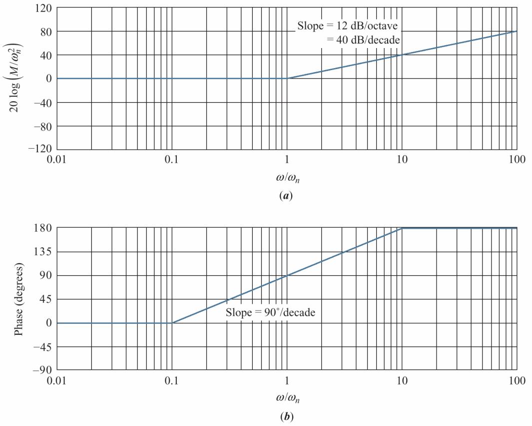


Figure 10.13.

Bode Plots for $G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$: Phase

- From previous slide, we know that we start with phase of 0^o at low frequencies, and end at phase of 180^o at high frequencies.

- Need to find phase at ω_n. Use:

$$G(j\omega) = s^2 + 2\zeta\omega_n s + \omega_n^2|_{s\to j\omega} \quad (4)$$

$$= (\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega \quad (5)$$

- At $\omega = \omega_n$, we get $G(j\omega) = j2\zeta\omega_n^2$, thus phase of 90^o.
- To draw phase curve:
  1. Start graph at $0.01\omega_n$ at phase of zero.
  2. At $0.1\omega_n$, switch to line with slope of $+90^o$ per decade.
  3. At $10\omega_n$, switch to horizontal line at 180^o.

Corrections for Second-Order Bode Plots

- Unlike first-order Bode plots, those of second-order systems can differ greatly from the approximations for certain values of ζ.

- From Eqn (5) on previous slide, we can derive magnitude and phase equations for $G(j\omega)$:

$$M(\omega) = \sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$
$$\phi(\omega) = tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}$$

- For magnitude, can make a $+20\log 2\zeta$ correction at the break frequency, ω_n.

Corrections for Second-Order Bode Plots - Magnitude

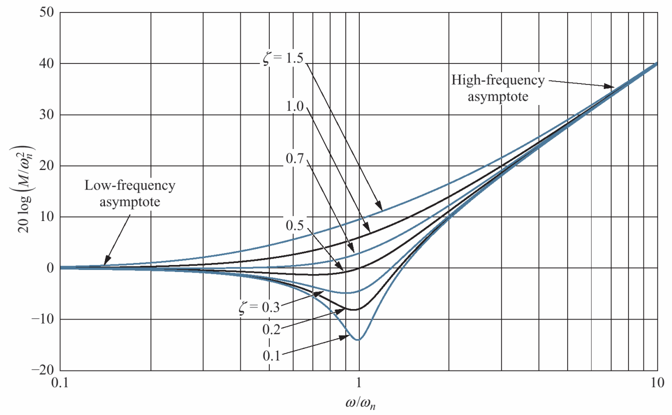


Figure 10.14.

Corrections for Second-Order Bode Plots - Phase

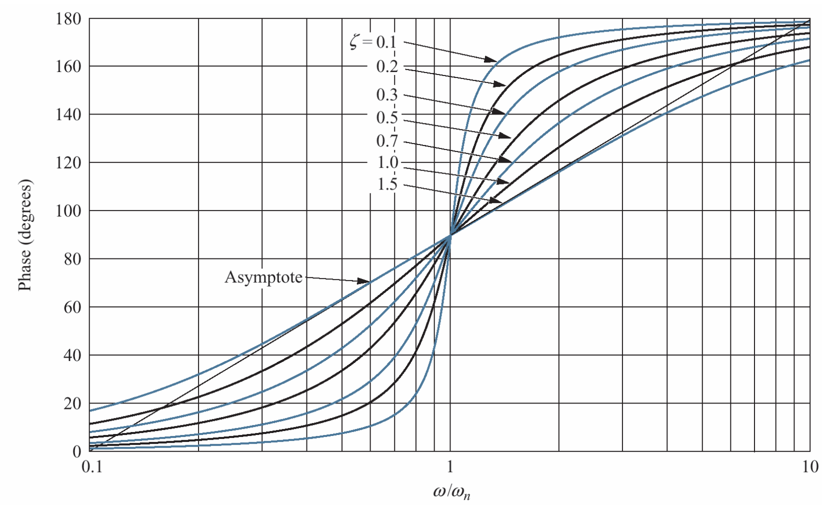


Figure 10.15.

Bode Plots for $G(s) = 1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$

- When ω approaches zero, we get $G(j\omega) \approx \frac{1}{\omega_n^2} = \frac{1}{\omega_n^2}\angle 0^o$ with magnitude response

$$20 \log M = 20 \log \frac{1}{\omega_n^2}$$

- At high frequencies, we get $G(j\omega) \approx \frac{1}{(j\omega)^2} = \frac{-1}{\omega^2}$ $= \frac{1}{\omega^2}\angle - 180^o$ where

$$20 \log M = 20 \log \frac{1}{\omega^2} = -40 \log \omega$$

- This is a straight line with a -40dB/dec slope and break frequency ω_n rad/s.

Bode Plots for $G(s) = 1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$ - II

- To draw magnitude approximation, start with a horizontal line at $20 \log \frac{1}{\omega_n^2}$, and at frequency ω_n, switch to a line with slope -40db/decade.
- Can make a $-20 \log 2\zeta$ correction at the break frequency, ω_n.

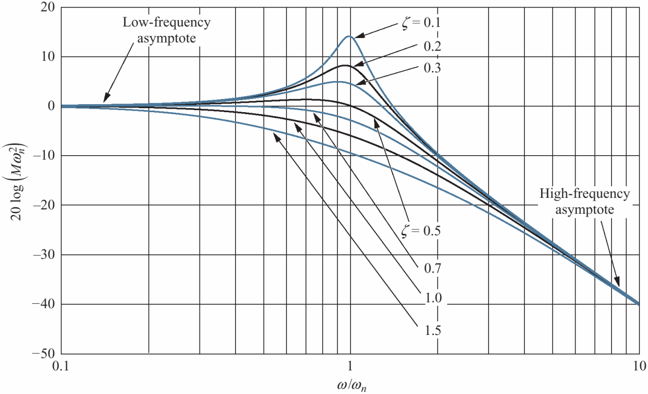


Figure 10.16.

Bode Plots for $G(s) = 1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$ - Phase

- To draw phase curve, start graph at $0.01\omega_n$ at 0^o, then at $0.1\omega_n$, switch to line with slope of -90^o per decade.

- Then at $10\omega_n$, switch to horizontal line at -180^o.

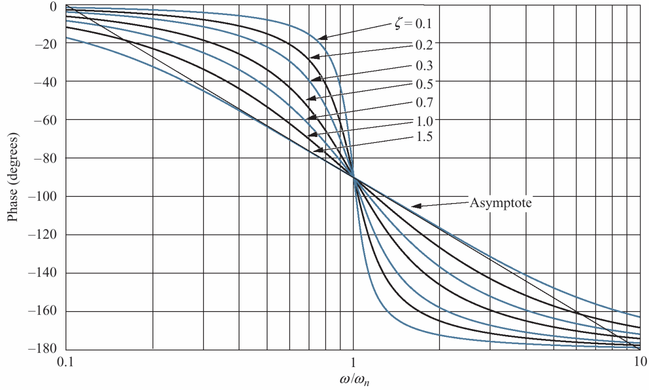


Figure 10.17.

Bode Plots for Ratio of 1st- and 2nd Order Factors

- Draw bode plots for system where

$$G(s) = \frac{(s+3)}{(s+2)(s^2+2s+25)}$$

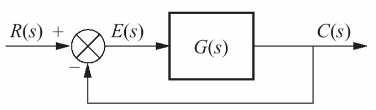


Figure 10.10.

Bode Plots for Ratio of 1st-2nd Order Factors - $M(\omega)$

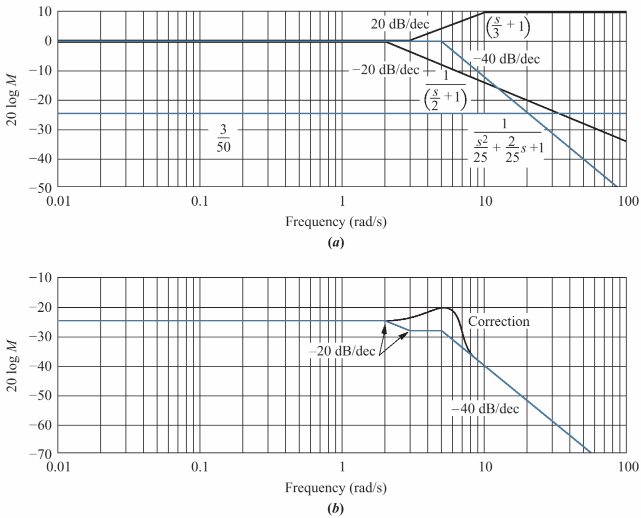


Figure 10.18.

Bode Plots for Ratio of 1st-2nd Order Factors - $\phi(\omega)$

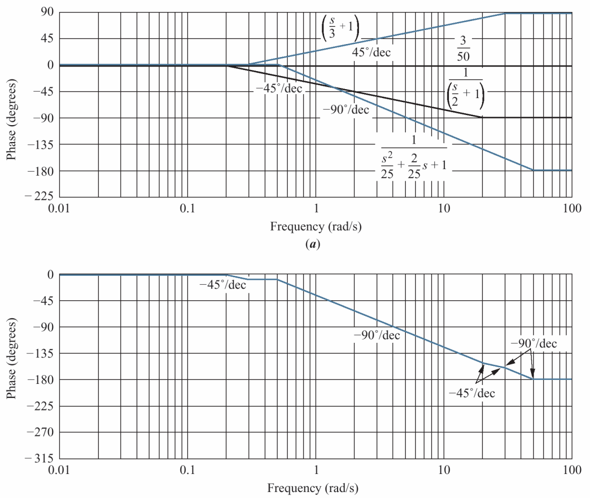


Figure 10.19.

Gain and Phase Stability Margins

- For a specific value of K and s, we know a closed loop pole exists when

$$1 + KG(s)H(s) = 0$$

- This is equivalent to when $KG(s)H(s) = -1 = 1\angle(2k+1)180^o \qquad k = 0, \pm1, \pm2, \ldots$
- If we set $s = j\omega$, we have imaginary roots when

$$|KG(j\omega)H(j\omega)| = 1 \qquad \angle KG(j\omega)H(j\omega) = (2k+1)180^o$$

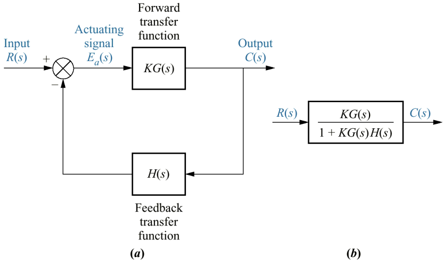


Figure 8.1.

Gain and Phase Stability Margins II

- From root locus, typically as we increase the gain K, the system changes from stable to unstable or vice versa when we cross the imaginary axis.

Case 1: If system becomes unstable as K increases, the stability condition is

$$|KG(j\omega)H(j\omega)| < 1 \qquad \angle KG(j\omega)H(j\omega) = (2k+1)180^o$$

Case 2: If system becomes stable as K increases, the stability condition is

$$|KG(j\omega)H(j\omega)| > 1 \qquad \angle KG(j\omega)H(j\omega) = (2k+1)180^o$$

Gain Margin

- To provide a measure of how stable the system, we use the concept of gain margin.
- Gain margin is how much we would have to change the frequency response magnitude curve to reach 0dB when the phase is $(2k+1)180^o$ (typically $\pm 180^o$).

- First, determine frequency ω_{G_M} which is when the phase is $(2k+1)180^o$.
- Then determine the value M of the magnitude at frequency ω_{G_M}.
- The gain margin is thus $G_M = 0 - M$ dB.

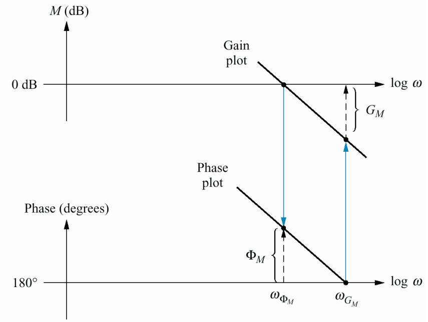


Figure: 10.37

Phase Margin

- We also use phase margin to provide a measure of how stable the system.
- Phase margin is how to change frequency response phase curve at the frequency that magnitude diagram is at 0dB, to achieve a phase of $(2k+1)180^o$ (typically $\pm 180^o$).

- First, determine frequency ω_{ϕ_M} which is when the gain is 0dB.
- Then determine the value ϕ of the phase at frequency ω_{ϕ_M}.
- The phase margin is thus $\phi_M = |\phi + 180^o|$ ($\phi < 0$) or $\phi_M = |180^o - \phi|$ ($\phi >= 0$).

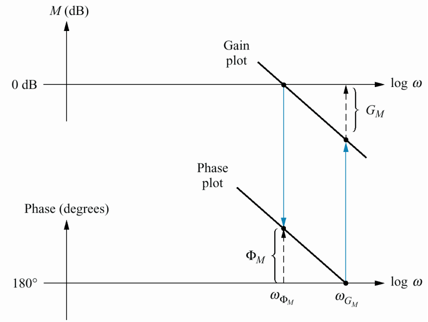
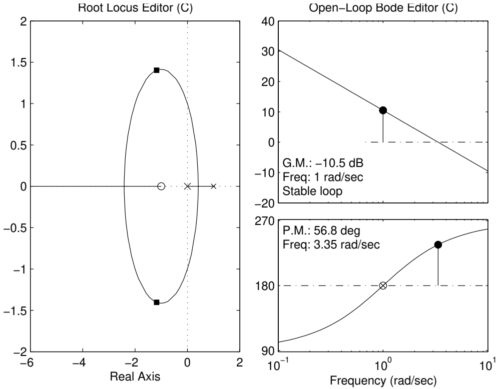


Figure: 10.37

Gain and Phase Stability Margins e.g.

- With $G(s)H(s) = \frac{K(s+1)}{s(s-1)}$, we see from root locus we have Case 2.
- From plot, we see that for $K = 3.35$ we get $\omega_{G_M} = 1$ rad/s and $M = 10.5$dB. Thus $G_M = 0 - 10.5 = -10.5$dB.
- Also, $\omega_{\phi_M} = 3.35$ rad/s and $\phi = 236.8$. Thus, $\phi_M = |180^o - 236.8| = 56.8^o$.



System Bandwidth

- The bandwidth of a system (ω_{BW}) is the maximum frequency that a system will be able to track a sinusoidal satisfactorily.

- By "satisfactorily" tracking, we mean that the power from input to output is reduced by no more than 50% relative to the DC value.

- As power varies by the square of amplitude of signal, this translates to

$$\left|\frac{G(j\omega_{BW})}{G(j0)}\right|^2 = \frac{1}{2}$$

- We thus have $\left|\frac{G(j\omega_{BW})}{G(0)}\right| = \frac{1}{\sqrt{2}}$ or $20\log\frac{1}{\sqrt{2}} = -3$dB.

- Thus, ω_{BW} is the frequency at which the magnitude of the frequency response is -3dB below the DC value.

System Bandwidth e.g.

- Determine system bandwidth of system below.

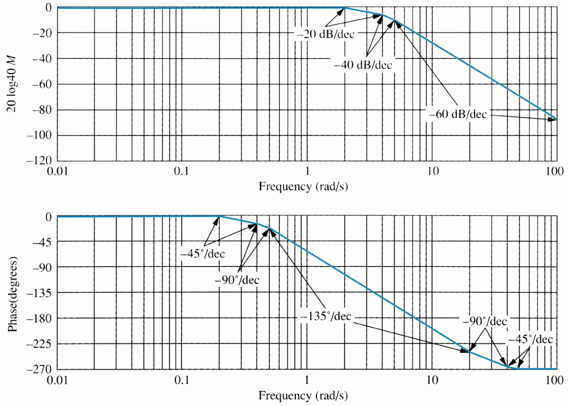


Figure 10.36.