More About Z and Inverse Z Transform

Z Transform Properties

- Linearity

  If $x(n) = a\ f_1(n) + bf_2(n)$, we have $X(z) = aF_1(z) + bF_2(z)$

- Time Shifting

$$\mathrm{Z}[x(t)] = \ X(z) \quad x(k-n) \leftrightarrow z^{-n}X(z)$$

$$x(k-1) \leftrightarrow z^{-1}X(z)$$

How to get partial fraction expression of $G(z) = \frac{z}{z^2-3z+2}$?

- $G(z) = \frac{z}{z^2-3z+2} \Rightarrow G(z) = \frac{z}{(z-1)(z-2)}$
- Write $G(z) = \frac{A}{(z-1)} + \frac{B}{(z-2)} = \frac{z}{(z-1)(z-2)}$, so we can solve A and B.

$$\frac{z}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)} = \frac{(A+B)z-(2A+B)}{(z-1)(z-2)}$$

So $(A+B) = 1$ and $(2A+B) = 0$

We got A=-1 and B=2

- We know $G(z) = \frac{-1}{(z-1)} + \frac{2}{(z-2)}$

Another Form of Z Transfer Function

- $G(z) = \frac{z}{(z-1)(z-2)}$
- We first get the partial fraction expression of $\frac{G(z)}{z} = \frac{1}{(z-1)(z-2)}$
- We get $\frac{G(z)}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$ ➔ $G(z) = \frac{-z}{z-1} + \frac{z}{z-2}$

On the previous page, we got $G(z) = \frac{-1}{(z-1)} + \frac{2}{(z-2)}$.

Are these G(z) we got both correct?

For $G(z) = \frac{z}{(z-1)(z-2)} = \frac{-z}{z-1} + \frac{z}{z-2} = \sum_{k=0}^{\infty}(-1 + 2^k)z^{-k}$,
We have $g(kT) = 2^k - 1$

- How about $G(z) = \frac{-1}{(z-1)} + \frac{2}{(z-2)}$?

- $G(z) = \frac{-1}{(z-1)} + \frac{2}{(z-2)} = z^{-1}\left[\frac{-z}{(z-1)} + \frac{2z}{(z-2)}\right]$

$G(z) = z^{-1}\sum_{k=0}^{\infty}\left(-1 + 2 \times 2^k\right)z^{-k} = \sum_{k=0}^{\infty}\left(-1 + 2 \times 2^k\right)z^{-(k+1)}$

$$\therefore G(z) = \sum_{k=0}^{\infty}\left(-1 + 2^{k+1}\right)z^{-(k+1)} = \sum_{k=1}^{\infty}\left(-1 + 2^k\right)z^{-k}$$

- So $g(kT) = 2^k - 1$, and $g(0) = 0$.

Find Z(e^{-t})➔Z(e^{-n})

We know that $Z(a^n) = \frac{z}{z-a}$

$$Z(e^{-n}) = Z\left[\left(e^{-1}\right)^n\right] = \frac{z}{z - e^{-1}}$$

Find Z[x(t)=t] ➔ Z(n) = n

$$Z(x(n) = n) = \sum_{n=0}^{\infty} n\, z^{-n} = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$Z(n) = \frac{1}{z}[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots]$$

$$Z(n) = \frac{1}{z}\left[\left(1 - \frac{1}{z}\right)^{-2}\right] = \frac{z}{(z-1)^2}$$

Examples

- Find inverse z-transform of each $\mathrm{E}(z)$

(1) $\mathrm{E1}(z) = \frac{0.5\,z}{(z-1)(z-0.6)}$

(2) $\mathrm{E2}(z) = \frac{0.5}{(z-1)(z-0.6)}$

(3) $E3(z) = \frac{0.5\,(z+1)}{(z-1)(z-0.6)}$

Using Power Series Method To Obtain Inverse Z-Transform of E1(z)

$$
\begin{array}{l}
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0.5z^{-1} + 0.8z^{-2} + 0.98z^{-3} + \cdots \\
\text{(i)}\ \ z^2 - 1.6z + 0.6 \,\big)\, 0.5z \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{0.5z - 0.8 + 0.3z^{-1}} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0.8 - 0.3z^{-1} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{0.8 - 1.28z^{-1} + \cdots} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0.98z^{-1} + \cdots
\end{array}
$$

Using Partial Fraction Expression Method to Solve Inverse Z-Transform of E1(z)

Need to know the sum of geometric series

$$\frac{z}{z-a} = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + a^4 z^{-4} + \cdots = \sum_{k=0}^{\infty} a^k z^{-k}$$

$$\frac{z}{z-1} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \cdots = \sum_{k=0}^{\infty} z^{-k}$$

- Step 1: $\mathrm{E1}(z) = \frac{0.5\,z}{(z-1)(z-0.6)} = \frac{1.25z}{z-1} - \frac{1.25z}{z-0.6}$

- Step 2: Write E1(z) in power series, so

$$\mathrm{e1}(k) = 1.25\left[\sum_{k=0}^{\infty} z^{-k} - \sum_{k=0}^{\infty} 0.6^k z^{-k}\right] = 1.25\sum_{k=0}^{\infty}(1-0.6^k)z^{-k}$$

- Step 3: $e1(\mathrm{k}) = 1.25 \times \ (1-0.6^k)$

Partial Fraction Expression

$\frac{0.5\,z}{(z-1)(z-0.6)} = z \times \frac{0.5}{(z-1)(z-0.6)}$, We consider $\frac{0.5}{(z-1)(z-0.6)}$ now, and set

$$\frac{0.5}{(z-1)(z-0.6)} = \frac{A}{(z-1)} + \frac{B}{(z-0.6)}$$

Hence, $Az - 0.6A + Bz - B = 0.5$. We know

$A + B = 0$ and $0.6A + B = -0.5$ ➔ A=1.25 and B=-1.25

Hence,

$$\frac{0.5\,z}{(z-1)(z-0.6)} = \frac{1.25z}{z-1} - \frac{1.25z}{z-0.6}$$

Let’s work directly on E1(z) using partial fraction Expression

$$\mathrm{E1}(z) = \frac{0.5\,z}{(z-1)(z-0.6)} = \frac{A}{(z-1)} + \frac{B}{(z-0.6)}$$

We get A=1.25, B=-0.75. Hence,

$$\frac{0.5\,z}{(z-1)(z-0.6)} = \frac{1.25}{(z-1)} - \frac{0.75}{(z-0.6)}$$

From previous page, we have

$$\frac{0.5\,z}{(z-1)(z-0.6)} = \frac{1.25z}{z-1} - \frac{1.25z}{z-0.6}$$

Are they the same?

Is $Z^{-1}\left[\frac{1.25}{(z-1)}-\frac{0.75}{(z-0.6)}\right]=Z^{-1}\left[\frac{1.25z}{(z-1)}-\frac{1.25z}{(z-0.6)}\right]$?

$$E1(z)=\frac{1.25}{(z-1)}-\frac{0.75}{(z-0.6)}=z^{-1}\left(\frac{1.25z}{z-1}-\frac{0.75z}{z-0.6}\right)$$

$$E1(z)=z^{-1}\left(1.25\sum_{k=0}^{\infty}z^{-k}-0.75\sum_{k=0}^{\infty}0.6^{k}z^{-k}\right)$$

$$E1(z)=1.25\sum_{k=0}^{\infty}z^{-k-1}-0.75\sum_{k=0}^{\infty}0.6^{k}z^{-k-1}$$

$$E1(z)=1.25\sum_{k=0}^{\infty}z^{-k-1}-0.75/0.6\sum_{k=0}^{\infty}0.6^{k+1}z^{-k-1}$$

$$E1(z)=1.25\sum_{k=1}^{\infty}z^{-k}-1.25\sum_{k=1}^{\infty}0.6^{k}z^{-k}=1.25\sum_{k=1}^{\infty}(1-0.6^{k})z^{-k}=1.25\sum_{k=0}^{\infty}(1-0.6^{k})z^{-k}$$

Obtain Inverse Z-Transform of E2(z)=$\frac{0.5}{(z-1)(z-0.6)}$

- Using Partial-Fraction Expression Method

$$E2(z) = \frac{0.5}{(z-1)(z-0.6)} = \frac{1.25}{z-1} - \frac{1.25}{z-0.6}$$

Given that $\mathrm{E2}(z) = z^{-1}E1(z) = 1.25z^{-1}\left[\sum_{k=0}^{\infty} z^{-k} - \sum_{k=0}^{\infty} 0.6^k z^{-k}\right]$.

$$z^{-1}\sum_{k=0}^{\infty} 0.6^k z^{-k} = \boldsymbol{z^{-1}} + \boldsymbol{0.6z^{-2}} + 0.6^2\boldsymbol{z^{-3}} + 0.6^3\boldsymbol{z^{-4}} + \cdots$$

(note that $e2(0) = 0$)

$$e2(k) = 1.25 \times \left[\sum_{k=1}^{\infty} z^{-k} - \sum_{k=1}^{\infty} 0.6^{(k-1)} z^{-k}\right]$$

So $e2(k) = 1.25\left(1 - 0.6^{k-1}\right) = 1.25 - 2.083 \times 0.6^k$ for $k \geq 1$

Alternatively $E2(z) = \frac{0.5}{(z-1)(z-0.6)} = z^{-1}E1(z)$

- Time shift property of Z-Transform $\mathrm{Z}(x(k-1)) = z^{-1}X(z)$

- We know from (1) that

$$x(k) = e1(\mathrm{k}) = 1.25 \times\ (1 - 0.6^{k})$$

- Hence,

$$x(k-1) = e2(\mathrm{k}) = 1.25 \times\ (1 - 0.6^{k-1})$$

Using Partial Fraction Expression Method to Solve Inverse Z-Transform of E3(z)

$$E3(z) = \frac{0.5\,(z+1)}{(z-1)(z-0.6)} = \frac{2.5}{z-1} - \frac{2}{z-0.6}$$

$$E3(z) = z^{-1}(\frac{2.5z}{z-1} - \frac{2z}{z-0.6})$$

$$e3(0) = 0$$

For $k \geq 1$,

$$e3(k) = 2.5 - 2 \times 0.6^{k-1} = 2.5 + \left(\frac{2}{0.6}\right) * 0.6\text{^}k$$

$$e3(k) = 2.5 - 3.33 \times 0.6^{k}$$

Solve Inverse Z-Transform of E3(z)

- $E3(z) = \frac{0.5\ (z+1)}{(z-1)(z-0.6)} = E1(z) + E2(z)$
- So

$$e3(0) = 0$$

For $k \geq 1$,

$$e3(\mathrm{k}) = e1(\mathrm{k}) + e2(\mathrm{k})$$

$$e3(k) = 1.25 \times \left(1 - 0.6^{k}\right) + 1.25\left(1 - 0.6^{k-1}\right)$$

$$e3(k) = 1.25 \times \left(1 - 0.6^{k}\right) + 1.25 - 2.083 \times 0.6^{k}$$

$$e3(k) = 2.5 - 3.33 \times 0.6^{k}$$

Find Z-Transform Equivalent of G(s)

- For the transfer functions $\frac{1}{s^2(s+1)}$, obtain the z-transform equivalents using partial fractions and s- and z-transform tables.

- Answer:

$$\frac{1}{s^2(s+1)} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$$

- So

$$Z\left[\frac{1}{s^2(s+1)}\right] = Z\left[\frac{-1}{s}\right] + Z\left[\frac{1}{s^2}\right] + Z\left[\frac{1}{s+1}\right]$$

$$= -\frac{z}{z-1} + \frac{Tz}{(z-1)^2} + \frac{z}{z-e^{-T}}$$

S- and Z-Transform Table

$x(t)$	$X(s)$	$X(z)$
1. $\delta(t) = \begin{cases} 1 & t=0, \\ 0 & t=kT, k\neq 0 \end{cases}$	1	1
2. $\delta(t-kT) = \begin{cases} 1 & t=kT, \\ 0 & t\neq kT \end{cases}$	e^{-kTs}	z^{-k}
3. $u(t)$, unit step	$1/s$	$\frac{z}{z-1}$
4. t	$1/s^2$	$\frac{Tz}{(z-1)^2}$
5. t^2	$2/s^3$	$\frac{T^2z(z+1)}{(z-1)^3}$
6. e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
7. $1-e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
8. te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
9. t^2e^{-at}	$\frac{2}{(s+a)^3}$	$\frac{T^2e^{-aT}z(z+e^{-aT})}{(z-e^{-aT})^3}$
10. $be^{-bt}-ae^{-at}$	$\frac{(b-a)s}{(s+a)(s+b)}$	$\frac{z[z(b-a)-(be^{-aT}-ae^{-bT})]}{(z-e^{-aT})(z-e^{-bT})}$
11. $\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2-2z\cos \omega T+1}$
12. $\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos \omega T)}{z^2-2z\cos \omega T+1}$
13. $e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{(ze^{-aT}\sin \omega T)}{z^2-2ze^{-aT}\cos \omega T+e^{-2aT}}$

- Why there is no T in the z-transforms in this Table?

TABLE 2-3 *z*-Transforms

Sequence	Transform
$\delta(k-n)$	z^{-n}
1	$\frac{z}{z-1}$
k	$\frac{z}{(z-1)^2}$
k^2	$\frac{z(z+1)}{(z-1)^3}$
a^k	$\frac{z}{z-a}$
ka^k	$\frac{az}{(z-a)^2}$
$\sin ak$	$\frac{z \sin a}{z^2 - 2z\cos a + 1}$
$\cos ak$	$\frac{z(z-\cos a)}{z^2 - 2z\cos a + 1}$
$a^k \sin bk$	$\frac{az \sin b}{z^2 - 2az\cos b + a^2}$
$a^k \cos bk$	$\frac{z^2 - az\cos b}{z^2 - 2az\cos b + a^2}$

Find $f(k)$, for the $F(z) = \frac{z(z+2)(z+5)}{(z-0.4)(z-0.6)(z-0.8)}$

$$\frac{(z+2)(z+5)}{(z-0.4)(z-0.6)(z-0.8)} = \frac{A}{(z-0.4)} + \frac{B}{(z-0.6)} + \frac{C}{(z-0.8)}$$

➔

A(z^2-1.4z+0.48)+B(z^2-1.2z+0.32)+c(z^2-z+0.24)=z^2+7z+10

A+B+C=1

1.4A+1.2B+C=-7

0.48A+0.32B+0.24C=10

$$F(z) = 162 \times \frac{z}{(z-0.4)} - 364 \times \frac{z}{(z-0.6)} + 203 \times \frac{z}{(z-0.8)}$$

Hence,

$$f(k) = 162 \times (0.4)^{\mathrm{k}} - 364 \times (0.6)^{\mathrm{k}} + 203 \times (0.8)^{\mathrm{k}}$$

Discrete Transfer Function G(z)

A controller given by discrete transfer function:

$$G(z) = \frac{(z+1)(z-0.9512)}{(z-0.9039)(z-0.8616)}$$

Show that the controller may be realized in the form of a computer algorithm, given a controller output u(k) for an input signal e(k).

Answer:

$$G(z) = \frac{U(z)}{E(z)} = \frac{(z+1)(z-0.9512)}{(z-0.9039)(z-0.8616)}$$

$$\therefore \frac{U(z)}{E(z)} = \frac{z^2 + 0.0488z - 0.9512}{z^2 + 1.7655z - 0.7788} = \frac{1 + 0.0488z^{-1} - 0.9512z^{-2}}{1 + 1.7655z^{-1} - 0.7788z^{-2}}$$

So $U(z)\left(1 + 1.7655z^{-1} - 0.7788z^{-2}\right) = E(z)(1 + 0.0488z^{-1} - 0.9512z^{-2})$

By applying inverse Z-Transform to the above equation, the result is:

$$u(k) = e(k) + 0.0488e(k-1) - 0.9512e(k-2)$$
$$-1.7655u(k-1) + 0.7788u(k-2)$$

Given $F(z) = \frac{z+1}{z^2+5z+6}$, find $f(k) =$?

Answer:

$$F(z) = \frac{z+1}{z^2+5z+6} = z\text{^}(-1)[\frac{2z}{z+3} - \frac{z}{z+2}]$$

$$= 2\sum_{k=0}^{\infty}(-3)^k z^{-(k+1)} - \sum_{k=0}^{\infty}(-2)^k z^{-(k+1)}$$

$$f(k) = 2 \times (-3)^{k-1} - (-2)^{k-1}$$